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NOTE ON THE TERM MAXIMAL SUBGROUP.

By G. A. Miller.

The term *maximal subgroup*, or maximum subgroup, is the source of so much confusion on the part of the student of group theory that it seems worth while to consider the feasibility of replacing it by some other term. As such a term we would suggest *primary subgroup.* Instead of saying that the subgroup composed of all the substitions of a primitive group which omit a letter is maximal we should then say that this subgroup is primary, and thus associate the terms primary and primitive. Even if such a change of terms should not appear feasible a consideration of the objectional features of the term maximal subgroup may tend to reduce the confusion due to its use. This confusion is the more regretable because of the fact that it relates to elementary and fundamental properties of groups.

The size of a finite group is commonly measured by its order. If two groups have different orders, the one which has the larger order is said to be the larger group. This method of determining the relative magnitudes is also commonly used as regards subgroups. On the other hand, it is customary to call a subgroup a maximal subgroup, or a largest subgroup, even when the group contains subgroups whose orders are larger than that of this maximal subgroup. A necessary and sufficient condition that a subgroup is maximal is that it is not contained in a larger subgroup. In particular, the icosahedral group contains maximal subgroups of each of the following orders: 6, 10, 12.

It is possible to find a series of subgroups of any group G, beginning with any maximal subgroup G_1 and ending with the identity

$$G_1, G_2, \cdots, G_\lambda = 1,$$

such that the smallest subgroup of G which contains any of these subgroups besides G_1 is the one which precedes it in this series, while G_1 is not contained in any subgroup of G. The position of G_1 in this series seems to justify the term primary subgroup as a suggestive term for it. The subgroup which immediately precedes the identity in this series is of prime order. When the order of G is p^m, p being a prime number, $\lambda = m$.

In a group of prime power order every maximal subgroup is also a subgroup of maximal order and every maximal invariant subgroup is also of maximal order. Hence it might at first appear that the use of the term maximal subgroup as regards these groups would be unobjectionable.

That this is not the case results directly from the use of the term maximal abelian invariant subgroup.* If such a subgroup of a non-abelian group of order p^m is of order p^α, this group may contain larger invariant abelian subgroups as may be seen from the group of order 2^{10} defined as follows:

Let $s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8$ and $s_1, s_2, s_3, s_4, s_5, s_6, t_7, t_8$ be sets of generators of two abelian groups of order 2^8 and of type $(1, 1, 1, \cdots)$ and suppose that

$$t_7s_7t_7 = s_1s_7, \qquad t_7s_8t_7 = s_2s_8, \qquad t_8s_7t_8 = s_3s_7, \qquad t_8s_8t_8 = s_4s_8.$$

The group of order 2^{10} generated by the ten operators s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, t_7, t_8 has the interesting property that it contains two and only two abelian subgroups of order 2^8. A similar group can easily be constructed for every value of p and each of the groups thus constructed contains two and only two abelian subgroups of order p^8. These two subgroups are evidently both invariant and maximal abelian subgroups. They illustrate a statement made without proof in the *Finite Groups* by Miller, Blichfeldt, Dickson, 1916, page 126.

This group of order 2^{10} is transformed into itself by an operator t_9 which is of order 2 and satisfies the following conditions:

$$t_9s_2t_9 = s_3, \qquad t_9s_5t_9 = s_1s_5, \qquad t_9s_6t_9 = s_4s_6, \qquad t_9s_7t_9 = t_7, \qquad t_9s_8t_9 = t_8.$$

We thus obtain a group of order 2^{11} which has two conjugate abelian subgroups of order 2^8 but no invariant abelian subgroup whose order exceeds 2^7. The abelian subgroup of order 2^7 generated by s_1, s_2, s_3, s_4, s_5, s_6, s_7, t_7 is a maximal invariant abelian subgroup of the given group of order 2^{10} notwithstanding the fact that this group contains invariant abelian subgroups of larger order. As similar subgroups exist for all values of p it results that there are groups of order p^m, p being any prime number, which contain larger invariant abelian subgroups than some of their maximal invariant abelian subgroups. Hence it is clear that the term maximal subgroup is apt to lead to confusion even with respect to prime power groups.

It may be of interest to note in this connection that from the known theorem that every abelian subgroup of order p^α which is contained in a group of order p^m is found in $1 + kp$ abelian subgroups of order $p^{\alpha+1}$ whenever it is found in at least one such abelian subgroup† it results directly that every invariant abelian subgroup of order p^α is found in a number of invariant abelian subgroups of order $p^{\alpha+1}$ which is of the form $1 + kp$ whenever it is contained in at least one such subgroup. In particular, every invariant subgroup of any group of order p^m contains a primary or maximal invariant abelian subgroup which is invariant under the entire group. This theorem was proved in a different manner by Burnside in the article to which reference was made.

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* Cf. W. Burnside, Proceedings of the London Mathematical Society, vol. 13 (1914), p. 9.

† Miller, Messenger of Mathematics, vol. 36 (1907), p. 70.